NONADDITIVE ENTROPY
AND
NONEXTENSIVE STATISTICAL MECHANICS:
BASIC CONCEPTS AND RECENT APPLICATIONS

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Daejeon, September 2009
In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space [...]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. [...] For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).
THERMODYNAMICS

VLASOV EQUATION
BOLTZMANN KINETIC EQUATION
BBGKY HIERARCHY

FOKKER-PLANCK EQUATION

LANGEVIN EQUATION
MASTER EQUATION

LIOUVILLE EQUATION
VON NEUMANN EQUATION

STATISTICAL MECHANICS

MECHANICS (classical, quantum, relativistic …)

N → ∞

Theory of probabilities
+ Entropy functional

Braun and Hepp theorem

H theorem
**POSTULATE FOR THE ENTROPIC FUNCTIONAL**

<table>
<thead>
<tr>
<th>BG entropy ( (q = 1) )</th>
<th>( k \ln W )</th>
<th>Concave</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entropy ( S_q ) ( (q \text{ real}) )</td>
<td>( k \frac{W^{1-q} - 1}{1 - q} )</td>
<td>Extensive</td>
</tr>
<tr>
<td></td>
<td>( -k \sum_{i=1}^{W} p_i \ln p_i )</td>
<td>Lesche-stable</td>
</tr>
</tbody>
</table>

Finite entropy production per unit time

Pesin-like identity (with largest entropy production)

Composable

Topsoe-factorizable

Possible generalization of Boltzmann-Gibbs statistical mechanics

Emergent properties, Self-organization, Complexity

\[ q = \frac{1 + \delta}{2} \quad [\text{A. Robledo, Mol Phys 103 (2005) 3025}] \]

\[ q = \frac{\sqrt{9 + c^2} - 3}{c} \quad [\text{F. Caruso and C. T., Phys Rev E 78 (2008) 021101}] \]
EXTENSIVITY OF THE NONADDITIVE ENTROPY $S_q$
**ADDITIVITY:**


An entropy is **additive** if, for any two probabilistically independent systems $A$ and $B$,

\[ S(A + B) = S(A) + S(B) \]

Therefore, since

\[ S_q(A + B) = S_q(A) + S_q(B) + (1 - q) S_q(A) S_q(B) , \]

$S_{BG}$ and $S_q^{\text{Renyi}} (\forall q)$ are additive, and $S_q (\forall q \neq 1)$ is nonadditive.

**EXTENSIVITY:**

Consider a system $\Sigma \equiv A_1 + A_2 + \ldots + A_N$ made of $N$ (not necessarily independent) identical elements or subsystems $A_1$ and $A_2$, ..., $A_N$. An entropy is **extensive** if

\[ 0 < \lim_{N \to \infty} \frac{S(N)}{N} < \infty , \text{ i.e., } S(N) \propto N \ (N \to \infty) \]
Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics

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The Boltzmann–Gibbs–von Neumann entropy of a large part (of linear size $L$) of some (much larger) $d$-dimensional quantum systems follows the so-called area law (as for black holes), i.e., it is proportional to $L^{d-1}$. Here we show, for $d=1,2$, that the (nonadditive) entropy $S_q$ satisfies, for a special value of $q \neq 1$, the classical thermodynamical prescription for the entropy to be extensive, i.e., $S_q \propto L^d$. Therefore, we reconcile with classical thermodynamics the area law widespread in quantum systems. Recently, a similar behavior was exhibited in mathematical models with scale-invariant correlations [C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sci. U.S.A. \textbf{102} 15377 (2005)]. Finally, we find that the system critical features are marked by a maximum of the special entropic index $q$. 


SPIN ½ XY FERROMAGNET WITH TRANSVERSE MAGNETIC FIELD:

\[ \hat{\mathcal{H}} = - \sum_{j=1}^{N-1} \left[ (1 + \gamma)\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + (1 - \gamma)\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y + 2\lambda \hat{\sigma}_j^z \right] \]

- $|\gamma| = 1 \quad \rightarrow \quad \text{Ising ferromagnet}$
- $0 < |\gamma| < 1 \quad \rightarrow \quad \text{anisotropic XY ferromagnet}$
- $\gamma = 0 \quad \rightarrow \quad \text{isotropic XY ferromagnet}$

$\lambda \equiv \text{transverse magnetic field}$

$L \equiv \text{length of a block within a } N \rightarrow \infty \text{ chain}$

Using a Quantum Field Theory result in P. Calabrese and J. Cardy, JSTAT P06002 (2004) we obtain, at the critical transverse magnetic field,

\[ q_{\text{ent}} = \frac{\sqrt{9 + c^2} - 3}{c} \]

with \( c \equiv \text{central charge} \) in conformal field theory.

Hence

*Ising and anisotropic XY ferromagnets* \( \Rightarrow c = \frac{1}{2} \) \( \Rightarrow q_{\text{ent}} = \sqrt{37} - 6 \approx 0.0828 \)

and

*Isotropic XY ferromagnet* \( \Rightarrow c = 1 \) \( \Rightarrow q_{\text{ent}} = \sqrt{10} - 3 \approx 0.1623 \)

Summarizing, for a wide class of quantum problems, we know that

$$S_{BG}(N) \propto \ln L \propto \ln N \neq N \quad \text{for } d = 1 \text{ quantum chains}$$

$$\propto L \propto \sqrt{N} \neq N \quad \text{for } d = 2 \text{ bosonic systems}$$

$$\propto L^2 \propto N^{2/3} \neq N \quad \text{for } d = 3 \text{ black hole}$$

$$\propto L^{d-1} \propto N^{(d-1)/d} \neq N \quad \text{for } d\text{-dimensional bosonic systems} \quad (d > 1; \text{ area law})$$

$$\propto \frac{L^{d-1} - 1}{d - 1} \equiv \ln_{2-d} L \neq L^d \propto N \quad (d \geq 1) \quad \text{(NONEXTENSIVE!)}$$

For the same class of quantum problems, we expect

$$S_{q_{ent}}(N) \propto L^d \propto N \quad (d \geq 1; \quad q_{ent} \neq 1) \quad \text{(EXTENSIVE!)}$$

(which we have illustrated for $d = 1, 2$)

When entropy does not seem extensive


Everybody who knows about entropy knows that it is an extensive property, like mass or enthalpy. [...] Of course, there is more than that to entropy, which is also a measure of disorder. Everybody also agrees on that. But how is disorder measured? [...] So why is the entropy of a black hole proportional to the square of its radius, and not to the cube of it? To its surface area rather than to its volume?
<table>
<thead>
<tr>
<th>SYSTEMS</th>
<th>ENTROPY $S_{BG}$ (additive)</th>
<th>ENTROPY $S_q (q&lt;1)$ (nonadditive)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short-range interactions, weakly entangled blocks, etc</td>
<td><strong>EXTENSIVE</strong></td>
<td><strong>NONEXTENSIVE</strong></td>
</tr>
<tr>
<td>Long-range interactions (QSS), strongly entangled blocks, etc</td>
<td><strong>NONEXTENSIVE</strong></td>
<td><strong>EXTENSIVE</strong></td>
</tr>
</tbody>
</table>
\( q \)-GENERALIZATION OF THE CENTRAL LIMIT THEOREM
Optimization of

\[ S = - k \int dx \ p(x) \ \ln[p(x)] \]

with

\[ \int dx \ p(x) = 1 \]

and

\[ \langle E(x) \rangle \equiv \int dx \ p(x) \ E(x) = \text{constant} \]

yields

\[ p(x) = \frac{e^{-\beta E(x)}}{\int dy \ e^{-\beta E(y)}} \]

(Boltzmann-Gibbs distribution for thermal equilibrium)

Example: \( \langle x \rangle = 0 \) and \( \langle x^2 \rangle = \text{constant} \) yields

\[ p(x) = \frac{e^{-\beta x^2}}{\int dy \ e^{-\beta y^2}} \] (Gaussian distribution)
**q-GAUSSIANS:**

\[ p_q(x) \propto e_q^{-(x/\sigma)^2} \equiv \frac{1}{\left[1 + (q-1)(x/\sigma)^2\right]^{1/q}} \quad (q < 3) \]

CENTRAL LIMIT THEOREM

\( N^{1/[\alpha(2-q)]} \)-scaled attractor \( \mathbb{F}(x) \) when summing \( N \to \infty \) \( q \)-independent identical random variables with symmetric distribution \( f(x) \) with \( \sigma_Q \equiv \int dx \, x^2 [f(x)]^Q / \int dx \, [f(x)]^Q \) \( (Q = 2q-1, \, q_1 = \frac{1+q}{3-q}) \)

<table>
<thead>
<tr>
<th>( q = 1 ) [independent]</th>
<th>( q \neq 1 ) (i.e., ( Q = 2q-1 \neq 1 )) [globally correlated]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_Q &lt; \infty ) ( (\alpha = 2) )</td>
<td>( \mathbb{F}(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x) ), with same ( \sigma_Q ) of ( f(x) )</td>
</tr>
<tr>
<td>( \mathbb{F}(x) = \text{Gaussian } G(x) ), with same ( \sigma ) of ( f(x) )</td>
<td>( G_q(x) \sim \begin{cases} G(x) &amp; \text{if }</td>
</tr>
<tr>
<td>Classic CLT</td>
<td>with ( \lim_{q \to 1} x_c(q,2) = \infty )</td>
</tr>
<tr>
<td>( \sigma_Q \to \infty ) ( (0 &lt; \alpha &lt; 2) )</td>
<td>( \mathbb{F}(x) = L_{q,\alpha} ), with same (</td>
</tr>
<tr>
<td>( \mathbb{F}(x) = \text{Levy distribution } L_\alpha(x) ), with same (</td>
<td>x</td>
</tr>
<tr>
<td>\begin{align*} G(x) \ f(x) \sim C_x /</td>
<td>x</td>
</tr>
<tr>
<td>Levy-Gnedenko CLT</td>
<td>(intermediate regime)</td>
</tr>
<tr>
<td>( \text{with } \lim_{\alpha \to 2} x_c(1,\alpha) = \infty )</td>
<td>(distant regime)</td>
</tr>
</tbody>
</table>


S. Umarov, C. T., M. Gell-Mann and S. Steinberg cond-mat/0606038v2 and cond-mat/0606040v2 (2008)
SOME EXPERIMENTAL, OBSERVATIONAL
AND COMPUTATIONAL
VERIFICATIONS AND APPLICATIONS
Hydra viridissima:
A Upadhyaya, J-P Rieu, JA Glazier and Y Sawada, Physica A 293, 549 (2001)
**Dictyostelium discoideum (cells):**

A.M. Reynolds, Physica A (2009), in press

![Graphs showing velocity distributions for vegetative and starved cells]

**Equation:**

\[
P(v) = \frac{\Gamma \left( \frac{\alpha + 1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{\alpha}{2} \right)} \frac{v^\alpha}{v_a^\alpha} \left[ v_a^2 + v^2 \right]^\frac{\alpha}{2} \equiv \frac{P(0)}{\left[ 1 + (q-1) \beta v^2 \right]^{q-1}}
\]

- Vegetative: \( q = 5/3 \)
- Starved: \( q = 2 \)
LOGISTIC MAP: EDGE OF CHAOS

odd $2n$

$q = 1.63$

$\beta = 6.2$

even $2n$

$q = 1.70$

$\beta = 6.2$

U. Tirnakli, C. Beck and C. T.

U. Tirnakli, C. T. and C. Beck
COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:

PHYSICAL REVIEW A 67, 051402(R) (2003)

Anomalous diffusion and Tsallis statistics in an optical lattice

Eric Lutz

Sloan Physics Laboratory, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120
(Received 26 February 2003; published 27 May 2003)

We point out a connection between anomalous transport in an optical lattice and Tsallis’ generalized statistics. Specifically, we show that the momentum equation for the semiclassical Wigner function which describes atomic motion in the optical potential, belongs to a class of transport equations recently studied by Borland [Phys. Lett. A 245, 67 (1998)]. The important property of these ordinary linear Fokker-Planck equations is that their stationary solutions are exactly given by Tsallis distributions. An analytical expression of the Tsallis index \( q \) in terms of the microscopic parameters of the quantum-optical problem is given and the spatial coherence of the atomic wave packets is discussed.

(i) The distribution of atomic velocities is a \( q \)-Gaussian;

\[
(ii) \quad q = 1 + \frac{44E_R}{U_0}
\]

where \( E_R \equiv \) recoil energy

\( U_0 \equiv \) potential depth
Experimental and computational verifications in optical lattices:

Tunable Tsallis Distributions in Dissipative Optical Lattices

P. Douglas, S. Bergamini, and F. Renzoni

Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom

(Received 10 January 2006; published 24 March 2006)

We demonstrated experimentally that the momentum distribution of cold atoms in dissipative optical lattices is a Tsallis distribution. The parameters of the distribution can be continuously varied by changing the parameters of the optical potential. In particular, by changing the depth of the optical lattice, it is possible to change the momentum distribution from Gaussian, at deep potentials, to a power-law tail distribution at shallow optical potentials.
Experimental and computational verifications

(a) $W(p)$

$R^2 = 0.995$

(b) $q$

$q = 1 + \frac{44E_R}{U_0}$

(Computational verification: quantum Monte Carlo simulations)

(Experimental verification: Cs atoms)
Superdiffusion and Non-Gaussian Statistics in a Driven-Dissipative 2D Dusty Plasma

Bin Liu and J. Goree

Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA
(Received 1 June 2007; published 6 February 2008)

Anomalous diffusion and non-Gaussian statistics are detected experimentally in a two-dimensional driven-dissipative system. A single-layer dusty plasma suspension with a Yukawa interaction and frictional dissipation is heated with laser radiation pressure to yield a structure with liquid ordering. Analyzing the time series for mean-square displacement, superdiffusion is detected at a low but statistically significant level over a wide range of temperatures. The probability distribution function fits a Tsallis distribution, yielding $q$, a measure of nonextensivity for non-Gaussian statistics.

$\langle r^2 \rangle \propto t^\alpha$
Power-Law Distributions for a Trapped Ion Interacting with a Classical Buffer Gas

Ralph G. DeVoe

Physics Department, Stanford University, Stanford, California 94305, USA
(Received 3 November 2008; published 10 February 2009)

Classical collisions with an ideal gas generate non-Maxwellian distribution functions for a single ion in a radio frequency ion trap. The distributions have power-law tails whose exponent depends on the ratio of buffer gas to ion mass. This provides a statistical explanation for the previously observed transition from cooling to heating. Monte Carlo results approximate a Tsallis distribution over a wide range of parameters and have \textit{ab initio} agreement with experiment.
Devoe, Phys Rev Lett 102 (2009) 063001

\[ T(x) = \frac{T(0)}{\left[ 1+(q-1)\left( \frac{x}{\sigma} \right)^2 \right]^{\frac{1}{q-1}}} \]

FIG. 1 (color online). Monte Carlo distributions for a single $^{136}\text{Ba}^+$ ion cooled by six different buffer gases at 300 K ranging from $m_B = 4$ (left) to $m_B = 200$ (right). Note the evolution from Gaussian to power law (straight line) as the mass increases. The solid lines are Tsallis functions [Eq. (7)] with fixed $\sigma = 0.0185$ cm and the exponents of Table I.

<table>
<thead>
<tr>
<th>Buffer gas</th>
<th>$m_f/m_B$</th>
<th>$n$</th>
<th>$q_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>He</td>
<td>34.5</td>
<td>$&gt;60$</td>
<td>1.03</td>
</tr>
<tr>
<td>Ar</td>
<td>3.40</td>
<td>8.2</td>
<td>1.12</td>
</tr>
<tr>
<td>Kr</td>
<td>1.70</td>
<td>3.8</td>
<td>1.26</td>
</tr>
<tr>
<td>Xe</td>
<td>1.0</td>
<td>1.98</td>
<td>1.51</td>
</tr>
<tr>
<td>170</td>
<td>0.80</td>
<td>1.50</td>
<td>1.80</td>
</tr>
<tr>
<td>200</td>
<td>0.68</td>
<td>1.15</td>
<td>1.87</td>
</tr>
</tbody>
</table>
Generalized Spin-Glass Relaxation

R. M. Pickup, 1 R. Cywinski, 2, * C. Pappas, 3 B. Farago, 4 and P. Fouquet 4

1 School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, United Kingdom
2 School of Applied Sciences, University of Huddersfield, Huddersfield HD1 3DH, United Kingdom
3 Helmholtz Center Berlin for Materials and Energy, Glienickerstrasse 100, 14109, Berlin, Germany
4 Institut Laue Langevin, 6 rue Jules Horowitz, 38000 Grenoble, France

(Received 18 July 2008; published 4 March 2009)

Spin relaxation close to the glass temperature of CuMn and AuFe spin glasses is shown, by neutron spin echo, to follow a generalized exponential function which explicitly introduces hierarchically constrained dynamics and macroscopic interactions. The interaction parameter is directly related to the normalized Tsallis nonextensive entropy parameter q and exhibits universal scaling with reduced temperature. At the glass temperature q = 5/3 corresponding, within Tsallis’ q statistics, to a mathematically defined critical value for the onset of strong disorder and nonlinear dynamics.
SPIN RELAXATION IN SPIN GLASSES (NEUTRON SPIN ECHO):

\[ \phi(t) = \left[ 1 + \frac{q-1}{2-q} \left( \frac{t}{\tau} \right)^\beta \right]^{-\frac{2-q}{q-1}} \]

\[ \equiv \left[ 1 + (q_{rel} - 1) \left( \frac{t}{\tau} \right)^\beta \right]^{-\frac{1}{q_{rel} - 1}} \]

\[ q_{rel} \equiv \frac{1}{2-q} \]

Triangle for the entropic index $q$ of non-extensive statistical mechanics observed by Voyager 1 in the distant heliosphere

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Received 10 June 2005
Available online 11 July 2005
SOLAR WIND: Magnetic Field Strength


[Data: Voyager 1 spacecraft (1989 and 2002); 40 and 85 AU; daily averages]

\[ q_{sen} = -0.6 \pm 0.2 \]

\[ q_{rel} = 3.8 \pm 0.3 \]

\[ q_{stat} = 1.75 \pm 0.06 \]
Playing with additive duality \( (q \rightarrow 2 - q) \)
and with multiplicative duality \( (q \rightarrow 1 / q) \)
(and using numerical results related to the \( q \)– generalized central limit theorem)

we conjecture

\[
q_{rel} + \frac{1}{q_{sen}} = 2 \quad \text{and} \quad q_{stat} + \frac{1}{q_{rel}} = 2
\]

hence

\[
1 - q_{sen} = \frac{1 - q_{stat}}{3 - 2 q_{stat}}
\]

hence only one independent!

Burlaga and Vinas (NASA) most precise value of the \( q \)– triplet is

\[
q_{stat} = 1.75 = 7 / 4
\]

hence

\[
q_{sen} = -0.5 = -1 / 2 \quad (\text{consistent with } q_{sen} = -0.6 \pm 0.2 !)
\]

and

\[
q_{rel} = 4 \quad (\text{consistent with } q_{rel} = 3.8 \pm 0.3 !)
\]

C.T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc USA 102, 15377 (2005)
\[ \varepsilon_{\text{sen}} \equiv 1 - q_{\text{sen}} = 1 - \left( \frac{-1}{2} \right) = \frac{3}{2} \]
\[ \varepsilon_{\text{rel}} \equiv 1 - q_{\text{rel}} = 1 - 4 = -3 \]
\[ \varepsilon_{\text{stat}} \equiv 1 - q_{\text{stat}} = 1 - \frac{7}{4} = -\frac{3}{4} \]

We verify

\[ \varepsilon_{\text{stat}} = \frac{\varepsilon_{\text{sen}} + \varepsilon_{\text{rel}}}{2} \] (arithmetic mean!)

\[ \varepsilon_{\text{sen}} = \sqrt{\varepsilon_{\text{stat}} \varepsilon_{\text{rel}}} \] (geometric mean!)

\[ \varepsilon_{\text{rel}}^{-1} = \frac{\varepsilon_{\text{sen}}^{-1} + \varepsilon_{\text{stat}}^{-1}}{2} \] (harmonic mean!)

N.O. Baella (2008)
solar wind  
q-triplet  
q_{relaxation}  
q_{stat state}  
q_{sensitivity}  

\begin{align*}
\frac{1}{1 - q_{m/\alpha}^+} &= \frac{1}{1 - q_0} + \frac{m}{\alpha} \\
\frac{1}{1 - q_{m/\alpha}^-} &= \frac{1}{q_0 - 1} + \frac{m}{\alpha} \\
(0 < \alpha \leq 2; \ m = 0, \pm 1, \pm 2, \ldots)
\end{align*}

CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

\[ V(\vec{r}) \sim -\frac{A}{r^\alpha} \quad (r \to \infty) \quad (A > 0, \quad \alpha \geq 0) \]

integrable if \( \alpha / d > 1 \) (short-ranged)
non-integrable if \( 0 \leq \alpha / d \leq 1 \) (long-ranged)

\[ \int ds \quad (short-\text{ranged}) \]

\[ \int ds \quad (long-\text{ranged}) \]

EXTENSIVE SYSTEMS

- dipole-dipole
- dipole-monopole (tides)
- d-dimensional gravitation

NONEXTENSIVE SYSTEMS

- Newtonian gravitation

HMF (inertial XY model)
**d-DIMENSIONAL CLASSICAL INERTIAL XY FERROMAGNET:**

(We illustrate with the XY (i.e., $n=2$) model; the argument holds however true for any $n>1$ and any $d$-dimensional Bravais lattice)

$$H = K + V = \frac{1}{2I} \sum_{i=1}^{N} p_i^2 + \frac{J}{\mathcal{A}} \sum_{i,j} \frac{1 - \cos(\vartheta_i - \vartheta_j)}{r_{ij}^\alpha} \quad (I > 0, \ J > 0)$$

with \( \mathcal{A} \equiv \sum_{j=1}^{N} r_{ij}^{-\alpha} \propto \begin{cases} N^{1-\alpha/d} & \text{if } 0 \leq \alpha / d < 1 \\ \ln N & \text{if } \alpha / d = 1 \\ \text{constant} & \text{if } \alpha / d > 1 \end{cases} \)

and periodic boundary conditions.

*[The HMF model corresponds to $\alpha / d = 0$]*

A $q$-EXPONENTIAL DISTRIBUTION FOR THE ENERGY IMPLIES A $q$-GAUSSIAN DISTRIBUTION IN THE VELOCITY?

$$H(p_1,..., p_N, \vartheta_1,..., \vartheta_N) = K + V = \frac{1}{2I} \sum_{i=1}^{N} p_i^2 + \frac{J}{\mathcal{A}} \sum_{i,j} \frac{1 - \cos(\vartheta_i - \vartheta_j)}{r_{ij}^\alpha} \quad (I > 0, \ J > 0)$$

with $\mathcal{A} \equiv \sum_{j=1}^{N} r_{ij}^{-\alpha} \propto \begin{cases} 
N^{1-\alpha/d} & \text{if } 0 \leq \alpha / d < 1 \\
\ln N & \text{if } \alpha / d = 1 \\
n\text{constant} & \text{if } \alpha / d > 1 
\end{cases}$

and periodic boundary conditions.

[The HMF model corresponds to $\alpha / d = 0$]

$$P_N(p_1,..., p_N, \vartheta_1,..., \vartheta_N) \propto e^{-\beta H(p_1,..., p_N, \vartheta_1,..., \vartheta_N)} \Rightarrow$$

$$P_N(p_1) = \int \ldots \int dp_2...dp_N d\vartheta_1...d\vartheta_N \quad P(p_1,..., p_N, \vartheta_1,..., \vartheta_N) \propto e^{-\beta p_1^{2/2N}}$$

Numerically yes! ($\forall q \geq 1$)
$\ln_{1.5} \left[ \frac{P(p_1)}{P(0)} \right]$
HMF MODEL

\[ \lim_{\substack{\alpha \to \infty \\ N \to \infty}} \lim_{\substack{t \to \infty \\ N \to \infty}} \frac{5 - 3(\alpha / d)}{3 - (\alpha / d)} = \lim_{\substack{\alpha \to \infty \\ N \to \infty}} \lim_{\substack{t \to \infty \\ N \to \infty}} \frac{3 - 2(\alpha / d)}{2 - (\alpha / d)} = \lim_{\substack{\alpha \to \infty \\ N \to \infty}} \lim_{\substack{t \to \infty \\ N \to \infty}} \frac{7 - 5(\alpha / d)}{5 - 3(\alpha / d)} \]
Power law statistics and stellar rotational velocities in the Pleiades

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¹ Universidade Federal do Rio Grande do Norte, UFRN, Departamento de Física - C. P. 1641, Natal, RN, 59072-970, Brazil
² Universidade do Estado do Rio Grande do Norte - 59610-210, Mossoró, RN, Brasil

Maxwellian conjecture (C. T., 2009)
Full bibliography (regularly updated):

http://tsallis.cat.cbpf.br/biblio.htm

2859 articles by 2201 scientists from 64 countries

[22 September 2009]
<table>
<thead>
<tr>
<th>Country</th>
<th>Manuscripts</th>
<th>Country</th>
<th>Manuscripts</th>
<th>Country</th>
<th>Manuscripts</th>
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<td>USA</td>
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64 COUNTRIES  2201 SCIENTISTS

[Updated 22 September 2009]